The complexity of tropical polynomials and mean payoff games

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Min-plus Semiring

Min-plus semiring (tropical semiring):

$$(T, \oplus, \odot),$$

where T is $\mathbb{R}$ or $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, or $\mathbb{Z}$, or $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$,

$$x \oplus y = \min\{x, y\},$$

$$x \odot y = x + y.$$

Min-plus Linear Polynomials

Min-plus linear polynomial:

$$a_1 \odot x_1 \oplus \ldots \oplus a_n \odot x_n \text{ or}$$
$$\min(a_1 + x_1, \ldots, a_n + x_n).$$

$x \neq (\infty, \ldots, \infty)$ is a *root* if the minimum is attained at least twice. That is,

$$\exists k, l \quad a_k + x_k = a_l + x_l = \min_j (a_j + x_j).$$

This is also called *tropical equation*.

Example

Consider equation

$$0 \odot x \oplus 3 \odot y \oplus 2 \odot z \text{ or}$$
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Solutions: $(0 + t, -3, -2)$, $(0, -3 + t, -2)$, $(0, -3, -2 + t)$ for $t \geq 0$.

Also note that if (x, y, z) is a solution, then $(x + \alpha, y + \alpha, z + \alpha)$ is also a solution.

Min-plus Linear Equations

Min-plus linear equation:

$$a_1 \odot x_1 \oplus \ldots \oplus a_n \odot x_n = b_1 \odot x_1 \oplus \ldots \oplus b_n \odot x_n$$

or

$$\min(a_1 + x_1, \ldots, a_n + x_n) = \min(b_1 + x_1, \ldots, b_n + x_n).$$

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Consider

$$0 \odot x \oplus 1 \odot y = 2 \odot x \oplus 0 \odot y \text{ or}$$
$$\min(0 + x, 1 + y) = \min(2 + x, 0 + y).$$

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Consider

$$0 \odot x \oplus 1 \odot y = 2 \odot x \oplus 0 \odot y \text{ or}$$
$$\min(0 + x, 1 + y) = \min(2 + x, 0 + y).$$

Solutions: $(0, 0)$.
Here also if (x, y) is a solution, then $(x + \alpha, y + \alpha)$ is also a solution.

Origin

- Min-plus
  Combinatorial optimization, scheduling problems
- Tropical
  Algebraic geometry, mathematical physics.

Algebraic geometry

Consider the algebraic closure of the field of complex rational functions $\mathbb{C}(t)$. Its elements can be represented by Puiseux series locally at zero:

$$c_1 t^{d_1} + c_2 t^{d_2} + \ldots,$$

where $d_1 < d_2 < \ldots$ are rationals.
The order of the series above is d_1.
Consider polynomials in $\mathbb{C}(t)[x_1, \ldots, x_n]$. Then if $(a_1(t), \ldots, a_n(t)) \in \mathbb{C}(t)$ is a solution to some polynomial, then the sequence of orders is a solution to the corresponding tropical equation.

Systems of Min-plus and Tropical Equations

We consider *systems of tropical linear equations*

$$\min_{1 \le j \le n} \{a_{ij} + x_j\},\ 1 \le i \le m,$$

We call the set of common roots of polynomials of the system by *min-plus linear prevariety*.
We also consider systems of min-plus linear equations

$$\min_{1 \le j \le n} \{a_{ij} + x_j\} = \min_{1 \le j \le n} \{b_{ij} + x_j\},\ 1 \le i \le m.$$

The Main Problem

We are mostly interested in solving systems of tropical linear equations and systems of min-plus linear equations.

In the classical case there are polynomial time algorithms for this problem.

In min-plus case there are no polynomial time algorithms known.

It is however known that the solvability problem is in the complexity class $\mathrm{NP} \cap \mathrm{coNP}$.

Problems in $\mathrm{NP} \cap \mathrm{coNP}$

- Linear programming

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Bezem et al.(2010): mean payoff games $\sim$ Solvability of min-plus linear systems

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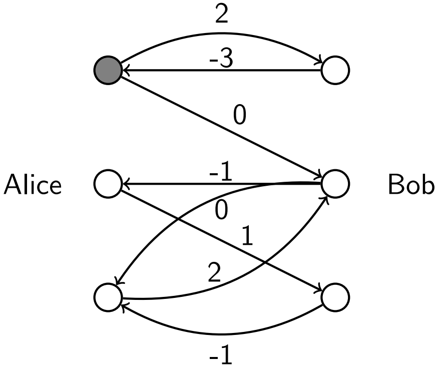
$\sim$ Bezem et al.(2010)

Grigoriev, P. (2012)

Mean Payoff Games

Ehrenfeucht, Mycielski ('79); Gurvich, Karzanov, Khachiyan ('88).
Two players, Alice and Bob, move a token over bipartite graph.



Alice tries to maximize the sum of edge labels, Bob tries to minimize it.
Game: $v_1, v_2, v_3, \ldots$
Value of the game: $\limsup_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \rho(v_i, v_{i+1})$.

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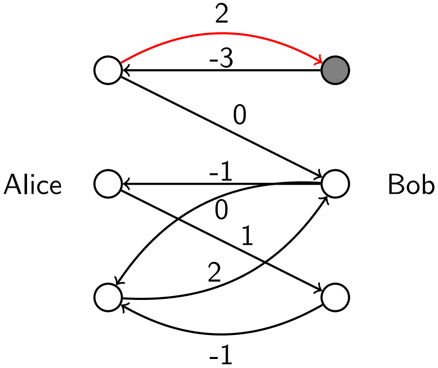

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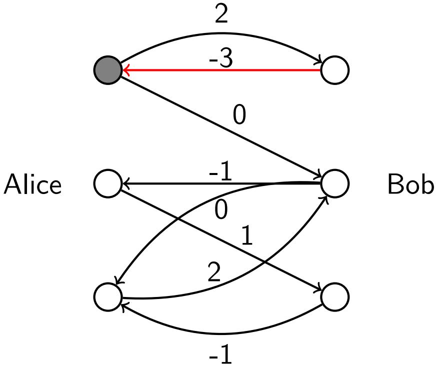

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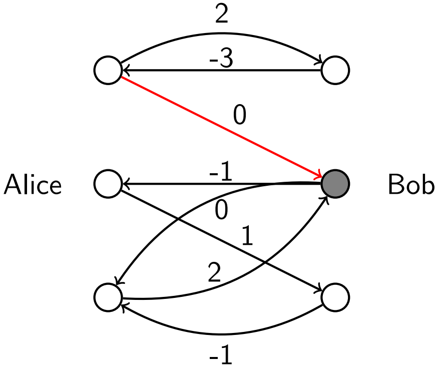

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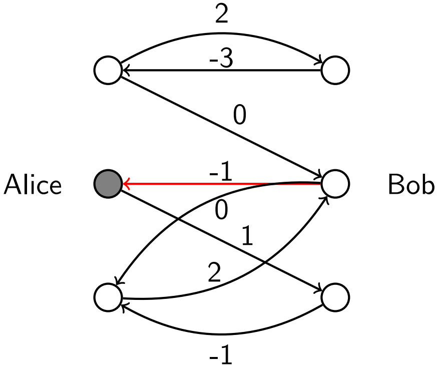

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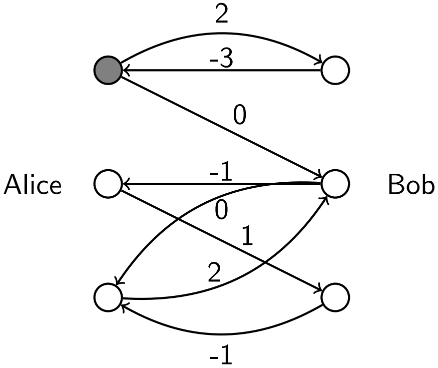

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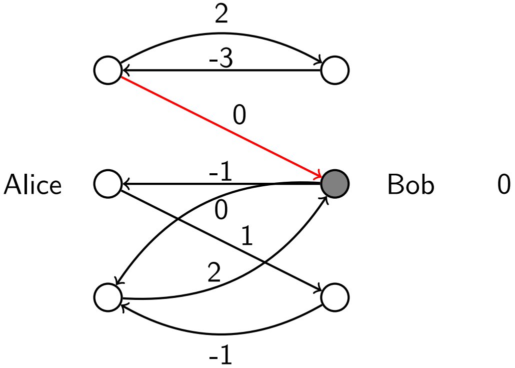

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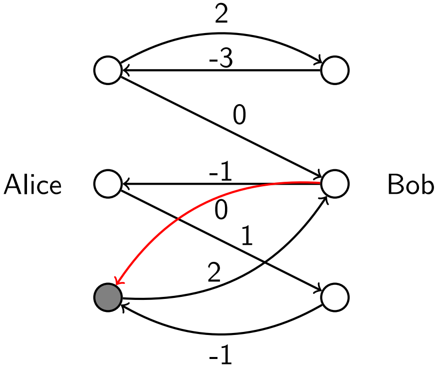

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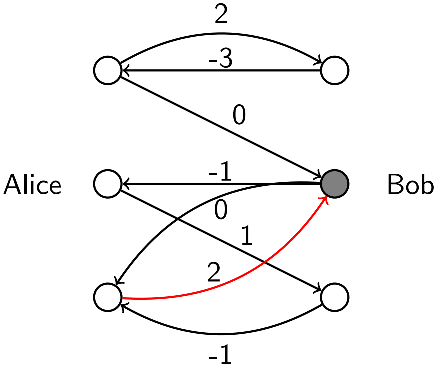

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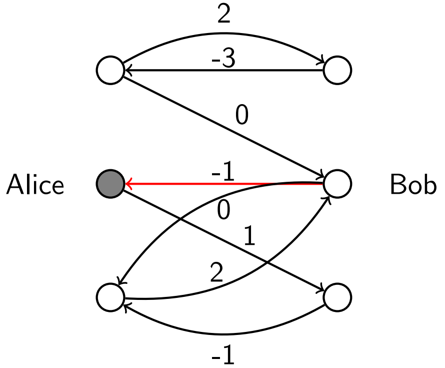

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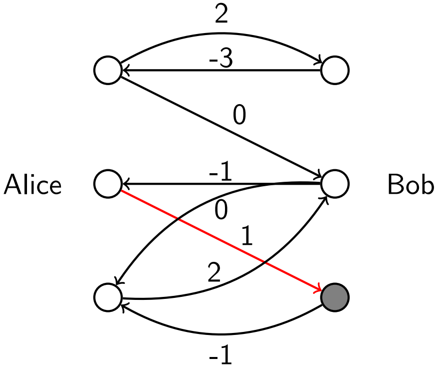

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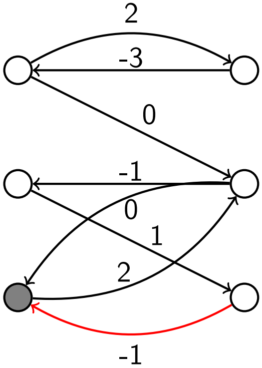
Alice  Bob $0 - 1 + 1 - 1$

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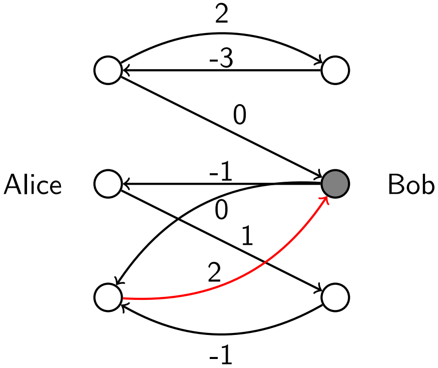

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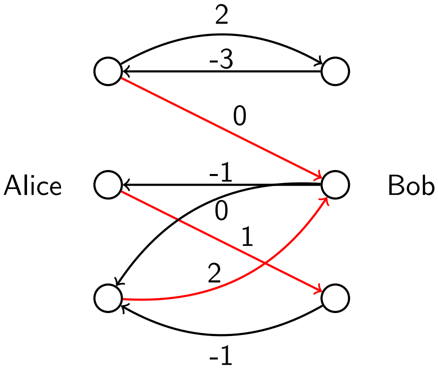

Value of the game: $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \rho(v_i, v_{i+1})$.
Alice wins if the value is positive. Otherwise Bob wins.
It is known that there are optimal positional strategies.

Mean Payoff Games

Mean payoff games problem: given a labeled graph decide whether Alice has a winning strategy.

The problem is in NP. For certificate we can take a winning strategy for Alice.

The problem is also in coNP. For certificate we can take a winning strategy for Bob.

Problems We Consider

Solvability problem TropSolv: Given an integer matrix $A \in \mathbb{Z}^{m \times n}$ decide whether the corresponding tropical linear system is solvable.

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Equivalence problem TropEquiv: Given two integer matrices $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{k \times n}$ decide whether the corresponding min-plus linear prevarieties are equal.

Dimension problem TropDim: Given an integer matrix $A \in \mathbb{Z}^{m \times n}$ and a number $k \in \mathbb{N}$ decide whether the dimension of the min-plus prevariety is at least k.

Note that the min-plus prevariety in $\mathbb{R}^n$ is a finite set of polytopes.

We also consider analogous problems over $\mathbb{Z}_\infty$ and also analogous problems for min-plus linear systems.

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We also consider analogous problems over $\mathbb{Z}_\infty$ and also analogous problems for min-plus linear systems.

All these problems are polynomial time solvable in classical case.

Results

	TropSolv	TropEquiv	TropDim
tropical	∼MPG[4]	∼MPG[4]	NP-complete[4]
min-plus	∼MPG[1,2]	∼MPG[3,4]	NP-complete[4]

[1] M. Bezem, R. Nieuwenhuis, and E. Rodríguez-Carbonell (2010)

[2] M. Akian, S. Gaubert, and A. Guterman (2012)

[3] X. Allamigeon, S. Gaubert, and R. D. Katz (2011)

[4] D. Grigoriev, V. Podolskii (2012)

All results are true both above $\mathbb{Z}$ and $\mathbb{Z}_\infty$.

Note that all problems in the first two columns are polynomial time equivalent.

Min-plus inequalities

We can consider the systems of linear min-plus inequalities.

But this is the same as equations.

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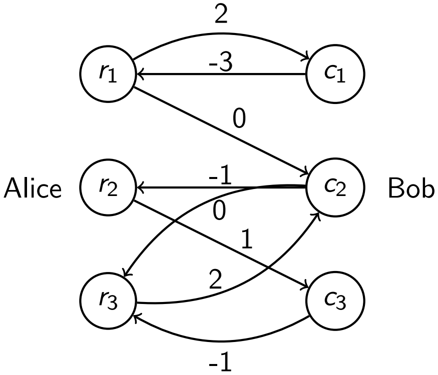
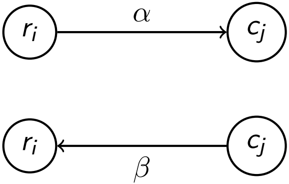
But this is the same as equations.

$L_1(x) = L_2(x)$ iff $L_1(x) \geq L_2(x)$ and $L_1(x) \leq L_2(x)$.

$L_1(x) \leq L_2(x)$ iff $L_1(x) = \min(L_1(x), L_2(x))$.

Min-plus and mean payoff games

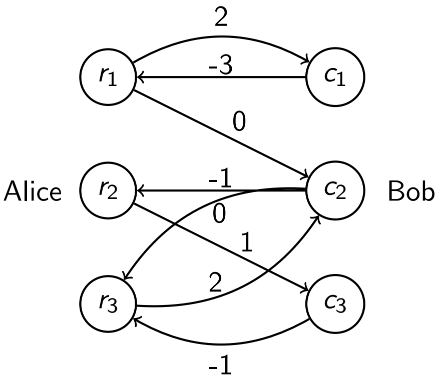


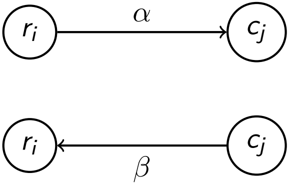


$$a_{ij} = -\alpha, b_{ij} = \beta$$

$$A = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \\ \begin{pmatrix} -2 & 0 & \infty \\ \infty & \infty & -1 \\ \infty & -2 & \infty \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \\ \begin{pmatrix} -3 & \infty & \infty \\ \infty & -1 & 0 \\ \infty & \infty & -1 \end{pmatrix} \end{array}$$

Min-plus and mean payoff games





$$a_{ij} = -\alpha, b_{ij} = \beta$$

$$\begin{pmatrix} -2 & 0 & \infty \\ \infty & \infty & -1 \\ \infty & -2 & \infty \end{pmatrix} \odot x \leq \begin{pmatrix} -3 & \infty & \infty \\ \infty & -1 & 0 \\ \infty & \infty & -1 \end{pmatrix} \odot x$$

Alice wins iff there is a solution to the system.

Tropical Solvability

Theorem
TropSolv *is polynomial time equivalent to mean payoff games.*

We will show that TropSolv is polynomial time equivalent to the solvability problem for the systems of min-plus equations.

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We will show that TropSolv is polynomial time equivalent to the solvability problem for the systems of min-plus equations.

For this we give reductions in both directions.

Solvability: Tropical $\rightarrow$ Min-plus

The following more strong connection is actually true.

Lemma

For a given system of linear tropical polynomials we can effectively construct an equivalent system of linear min-plus polynomials.

Solvability: Tropical $\rightarrow$ Min-plus

$$\min\{y_1, \ldots, y_n\} \text{ is attained at least twice}$$

iff

For all $i \in \{1, \ldots, n\}$

$$\min\{y_1, \ldots, y_n\} = \min\{y_1, ..., y_{i-1}, y_{i+1}, ..., y_n\}$$

iff

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$$\min\{y_1, ..., y_{i-1}, y_i, y_{i+1}, ..., y_n\} = \min\{y_1, ..., y_{i-1}, y_i + 1, y_{i+1}, ..., y_n\}$$

Solvability: Tropical $\rightarrow$ Min-plus

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$$\text{For all } i \in \{1, \ldots, n\}$$
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iff

$$\text{For all } i \in \{1, \ldots, n\}$$
$$\min\{y_1, ..., y_{i-1}, y_i, y_{i+1}, ..., y_n\} =$$
$$\min\{y_1, ..., y_{i-1}, y_i + 1, y_{i+1}, ..., y_n\}$$

Let $y_i = x_i + a_i$.

Solvability: Min-plus $\to$ Tropical

In the other direction we do not have such a tight connection. But we have it if we do not look at the neighborhood of infinity.

We say that two sets S and T in $\mathbb{R}^n$ are C-equal for an integer C, if $S \cap B(0, C) = T \cap B(0, C)$, where by $B(0, C)$ we denote the ball centered in 0 with the radius C.

Lemma

For any min-plus linear system $A \odot x \leq B \odot x$ over n variables and for arbitrary integer C there is a tropical linear system D over $2n$ variables and an injective linear transformation $H \colon \mathbb{R}^n \to \mathbb{R}^{2n}$ such that all solution of D lie in $Im(H)$ and the image of the solutions of (A, B) and the set of solutions of D are C-equal.

Technical Lemma

Lemma

Let $k \leq n$ and consider arbitrary vector $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$. Then for any $C \in \mathbb{Z}$ there is a tropical linear system $A \in \mathbb{Z}^{m \times n}$

$$A = \left(\begin{array}{c|c} \vec{a} & \\ \vec{a} & \geq C \\ \vec{a} & \end{array} \right),$$

where $m = n - k + 1$, such that for any solution of A and for any row the minimum is attained at least twice in the $\vec{a}$-part of the row.

Technical Lemma

Proof sketch.
$C = 99$

$$\left(\begin{array}{c|ccc} \vec{a} & 100 & 100 & 100 \\ \vec{a} & 99 & 100 & 100 \\ \vec{a} & 100 & 99 & 100 \\ \vec{a} & 100 & 100 & 99 \end{array}\right)$$

□

Solvability: Min-plus $\to$ Tropical

Lemma

For any min-plus linear system $A \odot x \leq B \odot x$ over n variables and for arbitrary integer C there is a tropical linear system D over $2n$ variables and an injective linear transformation $H\colon \mathbb{R}^n \to \mathbb{R}^{2n}$ such that all solution of D lie in $Im(H)$ and the image of the solutions of (A, B) and the set of solutions of D are C-equal.

Proof:

For each variable x_i of (A, B) we have two variables x_i and x_i' of D. For each i we apply Technical Lemma with $\vec{a} = (0, 0)$, $C = C$ to the variables x_i, x_i'. Denote the resulting system by T_i. In each its solution the variables x_i and x_i' are equal. We include systems T_i for all i into the system D.

Solvability: Min-plus → Tropical

Assuming $x_i = x_i'$ for all i we have

$$\min(a_1 + x_1, \dots, a_n + x_n) \le \min(b_1 + x_1, \dots, b_n + x_n)$$

iff for all $i = 1, \dots, n$

$$\min(a_1 + x_1, \dots, a_n + x_n) \le b_i + x_i$$

iff for all $i = 1, \dots, n$

$$\min(a_1 + x_1, a_1 + x_1', \dots, a_n + x_n, a_n + x_n', b_i + x_i)$$

is attained at least twice.

□

Even more strong connection is true over $\mathbb{Z}_\infty$.

Tropical Systems vs. Min-plus Systems

Thus we have that tropical linear systems and min-plus linear systems are in some sense "equivalent".

This equivalence is enough to prove that

1. the solvability problems for these systems are equivalent;
2. the equivalence problems for these systems are equivalent;
3. the dimensional problems for these systems are equivalent.

Note that our "equivalence" shows that geometrical structure of tropical linear systems and min-plus linear systems are almost the same.

Results

	TropSolv	TropEquiv	TropDim
tropical	∼MPG[4]	∼MPG[4]	NP-complete[4]
min-plus	∼MPG[1,2]	∼MPG[3,4]	NP-complete[4]

[1] M. Bezem, R. Nieuwenhuis, and E. Rodríguez-Carbonell (2010)

[2] M. Akian, S. Gaubert, and A. Guterman (2012)

[3] X. Allamigeon, S. Gaubert, and R. D. Katz (2011)

[4] D. Grigoriev, V. Podolskii (2012)

Dimension: Star Table

We move to the discussion of NP-completeness of $\textsc{TropDim}$.

Definition

Let A be a matrix of size $m \times n$. We associate with it the table A^* of the same size $m \times n$ in which we put the star $*$ to the entry (i, j) iff $a_{ij} = \min_k \{a_{ik}\}$ and we leave all other entries empty.

Note that $x = (x_1, \ldots, x_n)$ is a solution to the system A iff there are at least two stars in every row of the table $(\{a_{ij} + x_j\}_{ij})^*$.

Below we assume that in all tables we consider there are two stars in each row.

Dimension: Block Triangular Form

Definition

The block triangular form of size d of the matrix A is a partition of the set of rows of A into sets $R_1, R_2, \ldots, R_d$ (some of the sets R_i might be empty) and a partition of the set of columns of A into nonempty sets $C_1, \ldots, C_d$ with the following properties:

1. for every i each row in R_i has at least two stars in columns C_i in A^*;
2. if $1 \leq i < j \leq d$ then rows in R_i have no stars in columns C_j in A^*.

Dimension: Block Triangular Form

$$
A^* = \begin{array}{c}
\\
R_1 \\ \\ \\
R_2 \\ \\ \\
\vdots \\ \\ \\
R_d \\
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
 & C_1 & & & C_2 & & & \cdots & & & C_d &
\end{array}
\\
\left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & & & & & & & & & & \\
* & & * & & \emptyset & & & \emptyset & & & \emptyset & \\
 & * & * & & & & & & & & & \\
\hline
* & & & * & * & & & & & & & \\
* & * & & & * & * & & \emptyset & & & \emptyset & \\
 & & & * & & * & & & & & & \\
\hline
* & & & & & & * & & * & & & \\
* & & & * & * & * & * & * & & & \emptyset & \\
 & * & & & & & & * & * & & & \\
\hline
 & & & & * & & & & & & * & * \\
 & & * & & & * & & * & & * & * & \\
* & * & & & & & * & & & * & * &
\end{array}\right)
\end{array}
$$

Dimension: Combinatorial Characterization

Theorem

For a solution x of the tropical linear system A the local dimension of the system A in point x is equal to the maximal d such that there is a block triangular form of the matrix $\{a_{ij} + x_j\}_{ij}$ of size d.

Min-plus Dimension is NP-complete

Theorem

$\textsc{TropDim}$ *is* NP*-complete (both over* $\mathbb{Z}$ *and* $\mathbb{Z}_\infty$*).*

To show the containment in NP we can give as a certificate the point in which the dimension is achieved and the block triangular form.

To show the completeness we give a reduction from the vertex cover problem $\textsc{VertexCover}$: given a graph G and an integer k decide whether there is a set S of vertices of size at most k such that for each edge it least on of its ends is in S.

Higher degree

Min-plus monomial

$$M(x) = d_1x_1 + \ldots + d_nx_n,$$

where $d_i \geq 0$, d_i are integer.
Min-plus polynomial

$$p(x) = \min_i M_i(x),$$

where M_i are tropical monomials.
We consider min-plus polynomial equations of the form

$$p(x) = q(x).$$

Nullstellensatz

Theorem (Classical Nullstellensatz)

The system of polynomials $f_1, \ldots, f_m$ over algebraically closed field does not have a solution iff 1 lies in the ideal generated by $f_1, \ldots, f_m$.

Naive tropical reformulation is not true.
The min-plus system

$$x = 0, \quad x = 1$$

has no solution, but cannot generate $1 = 0$.

Min-plus Nullstellensatz

Theorem

The system of min-plus polynomial equations $f_1 = g_1, \ldots, f_m = g_m$ *has no solution iff we can construct an algebraic combination* $f = g$ *of them such that for each monomial* $M = x_1^{\odot d_1} \odot \ldots \odot x_n^{\odot d_n}$ *its coefficient in* f *is greater than its coefficient in* g*.*

There is also an analogous result for the tropical case.