Topological substitutions

Nicolas Bédaride, Arnaud Hilion, Timo Jolivet

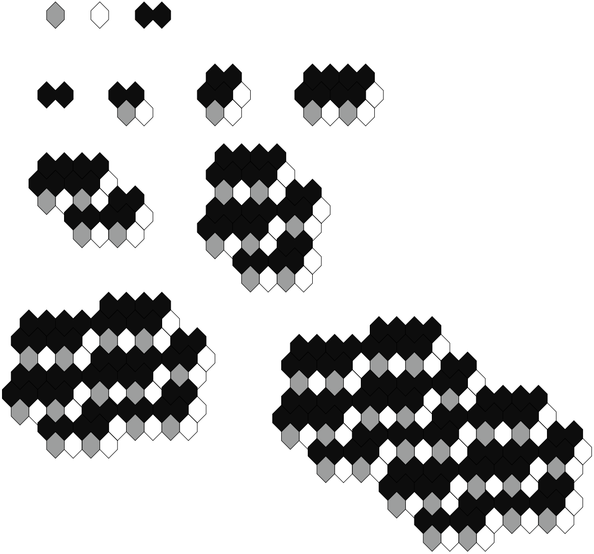


Figure: Iteration of the topological substitution.

Stepped plane

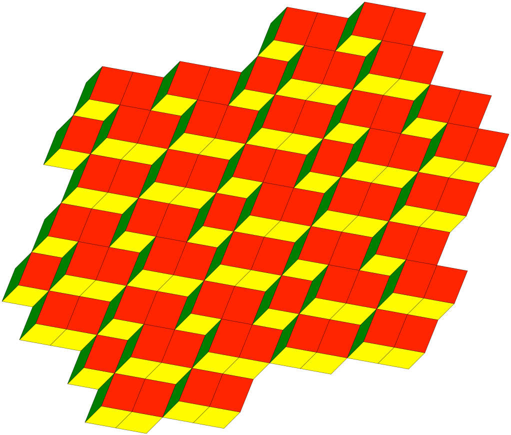


Figure: Aperiodic tiling of the stepped plane for the TRibonacci substitution.

Rauzy fractal

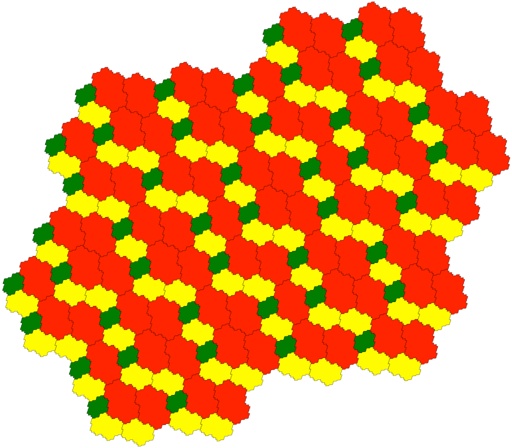


Figure: Aperiodic tiling with the Rauzy fractal for Tribonacci substitution.

Theorem

The three following tilings are mutually locally derivable

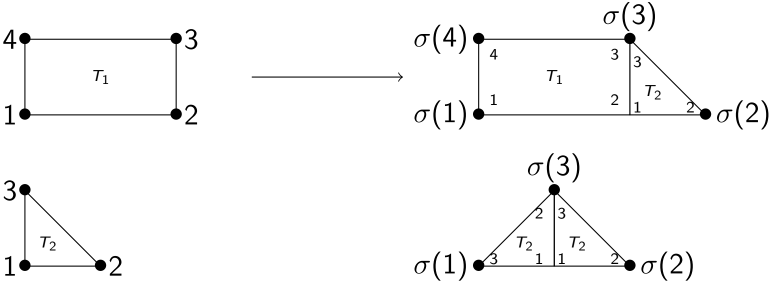
- *Tiling by the topological substitution.*
- *Tiling by the stepped surface.*
- *Tiling by the Rauzy fractal.*

Theorem

The sequence of renormalized patches $(\sigma^n(C))_{n\in\mathbb{N}}$ *is convergent for the Gromov-Hausdorff topology.*

Pre-substitution

- Cellular complex. Topological polygons $\mathcal{T} = \{T_1, \ldots, T_d\}$.
- Patches $\sigma(T_i), i = 1 \ldots d$.
- $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$.



A topological substitution is a pre-substitution such that

- can be iterated on a tile T
- $\sigma^n(T)$ has non empty core for each integer n.

Under hypothesis we can define

$$\sigma^\infty(T) = \bigcup_{k=0}^{\infty} \sigma^k(T).$$

By construction, the complex $\sigma^\infty(T)$ is homeomorphic to $\mathbb{R}^2$.

A tiling of $\mathbb{R}^2$ naturally defines a 2-complex.

Definition

A tiling of the plane $\mathbb{R}^2$ is **substitutive** if the labelled complex associated to it can be obtained by inflation from a topological substitution.

Main example

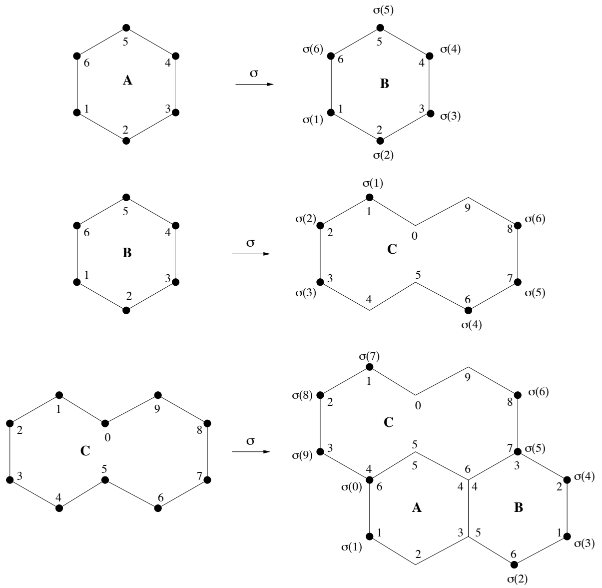


Figure: Tribonacci topological substitution

Tribonacci substitution

Morphism of the free monoid $\{a, b, c\}^*$

$$\sigma \begin{cases} a \mapsto ab \\ b \mapsto ac \\ c \mapsto a \end{cases}$$

a

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Fixed point $u = \sigma(u) = \lim_n \sigma^n(a)$

Tribonacci substitution

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The infinite word is called **Tribonacci word**.

Linear algebra

Matrix of the abelianization of σ is $M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

- Eigenvalue $\beta > 1$. Eigenvector v_β.
- Eigenvalue $\alpha \in \mathbb{C}, |\alpha| < 1$. Eigenspace: Plane P_α.
- Let us denote π_β the projection on P parallel to v_β.
- Let us denote I the abelianization map: $I : \{a, b, c\}^* \to \mathbb{Z}^3$.

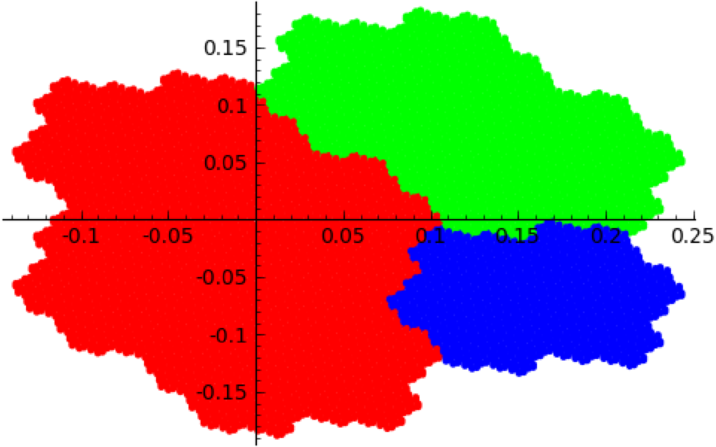
Rauzy fractal

Theorem (Rauzy)

There exists a compact set $\mathcal{R}$ in $P \sim \mathbb{R}^2$ and three subsets $\mathcal{R}_a, \mathcal{R}_b, \mathcal{R}_c$ such that

- $\mathcal{R}_j = \overline{\{\pi_\beta l(u_0 \dots u_{i-1}), u_i = j\}}$ *for* $j = a, b, c$.
- $\mathcal{R} = \overline{\{\pi_\beta l(u_0 \dots u_{i-1})\}}$
- *There exists a similitude* $h : \mathbb{R}^2 \to \mathbb{R}^2$*, such that:*
$$\begin{cases} \mathcal{R}_a = h\mathcal{R}_a \cup h\mathcal{R}_b \cup h\mathcal{R}_c \\ \mathcal{R}_b = h\mathcal{R}_a + \pi_\beta l(a) \\ \mathcal{R}_c = h\mathcal{R}_b + \pi_\beta l(a) \end{cases}$$
- *The compact set $\mathcal{R}$ tiles the plane in an aperiodic way.*

Rauzy fractal



Consequence

- The Rauzy fractal is solution of a **iterated function system**.

$$\mathcal{R} = h\mathcal{R} \bigcup h^2\mathcal{R} + \pi_\beta I_a \bigcup h^3\mathcal{R} + h\pi_\beta I_a + \pi_\beta I_a$$

- There is also a periodic tiling of the plane by this Rauzy fractal.

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Dual substitution

For $x \in \mathbb{Z}^3$ we define

$$(x, i^*) = \{x + e_i + \sum_{j \in \{a,b,c\}} \lambda_j e_j, \lambda_j \in (0;1), j \neq i\}$$

$$E_1^*(\sigma)(x, i^*) = \sum_{\substack{j \in \{a,b,c\}, \\ p.i.s = \sigma(j)}} (M_\sigma^{-1}(x - l(p)), e_j^*).$$

For example

- a belongs to $\sigma(a), \sigma(b), \sigma(c)$.
- b belongs to $\sigma(a)$.
- c belongs to $\sigma(a)$.

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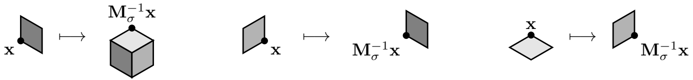
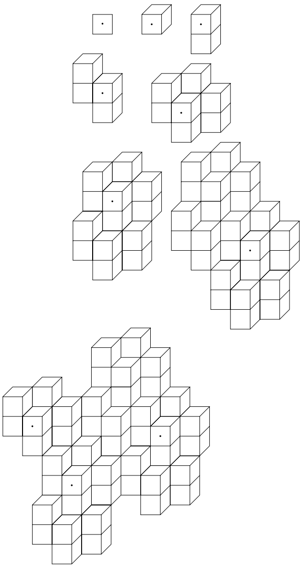


Figure: Dual substitution $E_1(\sigma)^*$ for the Tribonacci substitution

Iteration of the dual substitution



Theorem (Arnoux-Ito)

Define K as the set $\bigcup_{i\in\{a,b,c\}} (-e_i, i^*)$.

- *The sequence* $(\pi_P \circ E_1^*(\sigma)^n(K))_{n\in\mathbb{N}}$ *converges to a tiling of* $\mathcal{P}$.
- *The following limit exists for the Hausdorff convergence:*

$$\lim_n E_0(\sigma)^n(D_n)$$

where $D_n = \pi_\beta \circ E_1^*(\sigma)^n(K)$. *Moreover this limit is equal to the Rauzy fractal* $\mathcal{R}$.

- Link between Rauzy fractal and stepped surface: classical result.
- Link between Stepped surface and topological substitution.

Future

- Do the same think for another substitution.
- Find a topological substitution.
- The convergence will follow, and other properties.