On the zero-one k-law and its extensions

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Random graph $G(N, p)$

$$\Omega_N = \{(V = \{1, ..., N\}, E)\}, \ \ \mathcal{F}_N = 2^{\Omega_N},$$

$$\mathsf{P}_{N,p}(G) = p^{|E|}(1-p)^{C_N^2 - |E|}.$$

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First-order formulae
relational symbols $\sim, =$;
logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$;
variables $x, y, x_1, ...$;
quantifiers $\forall, \exists$.

Example: $\forall x \, \forall y \ \ (\neg(x=y) \Rightarrow (x \sim y))$.

$\mathcal{L}$ — a set of first-order properties.

G — an arbitrary graph; L_G — a property of containing G.
If $p \gg p_0 = N^{-1/\rho^{\max}(G)}$ then

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(L_G) = 1.$$

If $p \ll p_0 = N^{-1/\rho^{\max}(G)}$ then

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(L_G) = 0.$$

$N^{-1/\rho^{\max}(G)}$ — threshold function.
L_G is the first order property.

$\rho^{\max}(G) = \max_{H\subseteq G}\{\rho(H)\}$, $\rho(H) = \frac{e(H)}{v(H)}$.

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$\rho^{\max}(G) = \max_{H\subseteq G}\{\rho(H)\}$, $\rho(H) = \frac{e(H)}{v(H)}$.

$L(k)$ — a property of a graph to have chromatic number k.
There is no threshold function for $L(k)$.
$L(k)$ is the second order property.

- $p : \mathbb{N} \to [0,1]$.

Definition

The random graph obeys zero-one law if $\forall L \in \mathcal{L}$ one of the following properties hold

$$\lim_{N\to\infty} \mathsf{P}_{N,p(N)}(L) = 0, \quad \lim_{N\to\infty} \mathsf{P}_{N,p(N)}(L) = 1.$$

$\mathcal{P}$ is a set of functions p such that $G(N,p)$ obeys zero-one law.

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- $p = \text{const}$;
- $\forall \alpha > 0 \ \ \min\{p, 1-p\}N^{\alpha} \to \infty$ when $N \to \infty$;
- $p = N^{-\alpha}$, $\alpha \in (0,1]$;
- other cases.

Theorem [Y.V. Glebskii, D.I. Kogan, M.I. Liogonkii, V.A. Talanov, 1969; R. Fagin, 1976]

If $\min\{p, 1-p\}N^{\alpha} \to \infty$ when $N \to \infty$ for all $\alpha > 0$ then $p \in \mathcal{P}$.

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Theorem [J.H. Spencer, S. Shelah, 1988]

Let $p = N^{-\alpha}$, $\alpha \in (0, 1]$.

- If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $p \in \mathcal{P}$.
- If $\alpha \in \mathbb{Q}$ then $p \notin \mathcal{P}$.

In the first statement α *can be replaced with* $\alpha + o(1)$.

Theorem [J.H. Spencer, S. Shelah, 1988]

Let $p = p(N)$ be a function for which one of the following conditions holds:

- $N^{-1-1/l} \ll p \ll N^{-1-1/(l+1)}$ for some natural number l,
- $p \ll N^{-1}$ but $p = N^{-1-o(1)}$,
- $p \gg N^{-1}$ but $p = N^{-1+o(1)}$ and for every given pair of integers r and s such that $s \geq r - 1 \geq 0$ we have either $rNp - \log N - s \log\log N \to -\infty$ or $rNp - \log N - s \log\log N \to \infty$.

Then zero-one law holds.

$\mathcal{L}_k$ — a class of first-order properties defined by formulae with quantifier depth bounded by k.

The random graph $G(N,p)$ obeys zero-one k-law if $\forall L \in \mathcal{L}_k$

$$\lim_{N\to\infty} \mathsf{P}_{N,p(N)}(L) \in \{0,1\}.$$

$\mathcal{P}_k$ — class of functions $p = p(N)$ such that the random graph $G(N,p)$ obeys zero-one k-law.

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Example: $(\forall x \, \exists y \ \ (x \sim y)) \wedge (\forall x \, \exists y \ \ \neg(x \sim y));$

$$\forall x \, \exists y \, \exists z \ \ ((x \sim y) \wedge (\neg(x \sim z))).$$

Bounded quantifier depth. Infinite number of disjunctions and conjunctions.

$\mathcal{L}_k^\infty$ — a class of first-order properties defined by formulae with quantifier depth bounded by k, number of disjunctions and conjunctions can be infinite.

The random graph $G(N,p)$ obeys zero-one k-law with infinite sentences if $\forall L \in \mathcal{L}_k^\infty$

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$\mathcal{P}_k^\infty$ — a class of functions $p = p(N)$ such that the random graph $G(N,p)$ obeys zero-one k-law with infinite sentences.

Theorem [M. McArthur, 1997]

Let $p = N^{-\alpha}$, $k \in \mathbb{N}$, $k \geq 2$.

- If $0 < \alpha < \frac{1}{k-1}$ then $p \in \mathcal{P}_k^{\infty}$.
- If $\alpha = \frac{1}{k-1}$ then $p \notin \mathcal{P}_k^{\infty}$.

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Theorem 1 (zero-one k-law), 2010

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Theorem 2 (convergence k-law), 2012

Let $p = N^{-\alpha}$, $\alpha = \frac{1}{k-2}$, $k \in \mathbb{N}$, $k \geq 3$.
For any $L \in \mathcal{L}_k$ there exists $\lim\limits_{N \to \infty} \mathsf{P}_{N,p}(L)$.

Theorem 3, 2012

Let $p = N^{-\alpha}$, $k \in \mathbb{N}$, $k \geq 4$, $\mathcal{Q} = \{\frac{a}{b},\ a, b \in \mathbb{N}, a \leq 2^{k-1}\}$

- If $1 - \frac{1}{2^{k-1}} < \alpha < 1$, $\alpha \notin \mathcal{Q}$ then $p \in \mathcal{P}_k$.

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$$\left(1 - \frac{1}{2^k}, 1\right) \bigcup \left(1 - \frac{1}{2^k - 1}, 1 - \frac{1}{2^k}\right) \bigcup \ldots \bigcup$$

$$\left(1 - \frac{1}{2^{k-1} + 2^{k-2}}, 1 - \frac{1}{2^{k-1} + 2^{k-2} + 1}\right) \bigcup$$

$$\left(1 - \frac{1}{2^{k-1} + \frac{2^{k-1}-1}{2}}, 1 - \frac{1}{2^{k-1} + 2^{k-2}}\right) \bigcup \ldots \bigcup$$

$$\left(1 - \frac{1}{2^{k-1} + \frac{2^{k-1}}{3}}, 1 - \frac{1}{2^{k-1} + \frac{2^{k-1} - \left[\frac{2^{k-1}}{3}\right]}{2}}\right) \bigcup \ldots.$$

Theorem 4, 2012

Let $p = N^{-\alpha}$, $k \in \mathbb{N}$, $k \geq 4$.

- $\exists b_0(k) > 0 \quad \left(b > b_0(k), 0 < a < b, \alpha = \frac{a}{b} \Rightarrow p \in \mathcal{P}_k\right)$.

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Theorem 5, 2013

Let $k \in \mathbb{N}$, $k > 3$, $\tilde{\mathcal{Q}} = \{1, ..., 2^{k-1} - 2\}$.

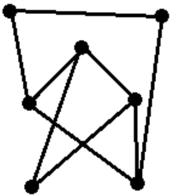
- $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$, $\beta \in \tilde{\mathcal{Q}} \Rightarrow p \notin \mathcal{P}_k$.

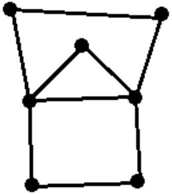
$\mathrm{EHR}(G, H, k)$

G, H — two graphs

k — number of rounds

Spoiler & Duplicator — two players





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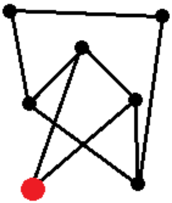

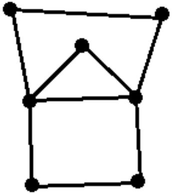

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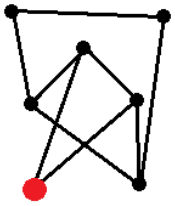

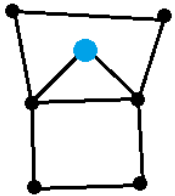

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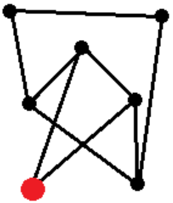

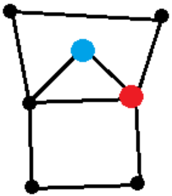

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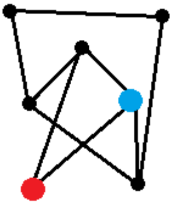

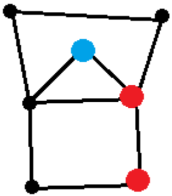

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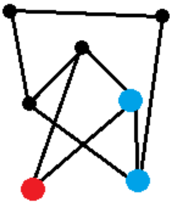

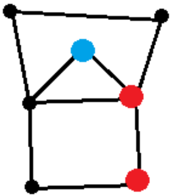

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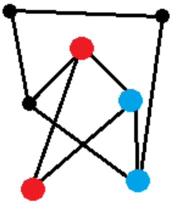

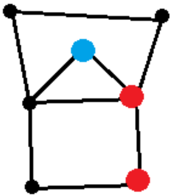

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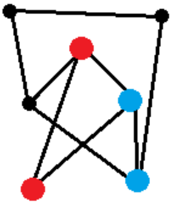

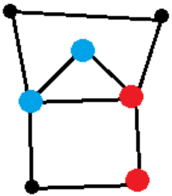

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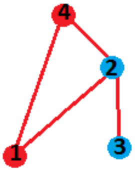

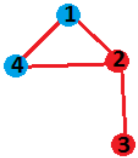

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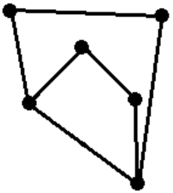
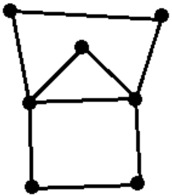
Theorem [A. Ehrenfeucht, 1960]

Let G, H be two graphs. For any first-order property L expressed by formula with quantifier depth bounded by a number k $G \in L \Leftrightarrow H \in L$ if and only if Duplicator has a winning strategy in the game $\mathrm{EHR}(G, H, k)$.

Example: $\exists x_1 \, \exists x_2 \, \exists x_3 \;\; ((x_1 \sim x_2) \wedge (x_2 \sim x_3) \wedge (x_1 \sim x_3))$.





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Corollary 1

- The zero-one law holds if and only if for any $k \in \mathbb{N}$ almost surely Duplicator has a winning strategy in the Ehrenfeucht game on k rounds.
- The random graph $G(N, p)$ obeys zero-one k-law if and only if almost surely Duplicator has a winning strategy in the Ehrenfeucht game on k rounds.

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Corollary 2

Let $\widetilde{\Omega}_N = \bigsqcup_{i=1}^{s} \Omega_N^i$, $\mathsf{P}_{N,p}(\widetilde{\Omega}_N) \to 1$, $N \to \infty$.
Let for any $N, M \in \mathbb{N}$, $i \in \{1, ..., s\}$, $G \in \Omega_N^i$, $H \in \Omega_M^i$ Duplicator have a winning strategy in the game $\mathrm{EHR}(G, H, k)$.
Then the convergence k-law holds.

The full level s extension property $S(s)$

$$\forall a, b \in \mathbb{Z}_+,\ a + b \leq s,\ v_1, ..., v_a, u_1, ..., u_b \in V$$
$$\exists x \in V \ \ (x \sim v_1) \wedge ... \wedge (x \sim v_a) \wedge (\neg((x \sim u_1) \wedge ... \wedge (x \sim u_b)))$$

- If for graphs G, H the full level $k-1$ extension property holds then Duplicator has a winning strategy in the game EHR(G, H, k).

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Proof

If $p = N^{-\alpha}$, $0 < \alpha < 1/(k-1)$, then for rather large N a probability of the full level $k-1$ extension property is greater than

$$1 - C_N^{k-1}(1-p^a(1-p)^b)^{N-k+1} > 1 - N^{k-1}\exp(-N^{\alpha(k-1)}(N-k)) \to 1.$$

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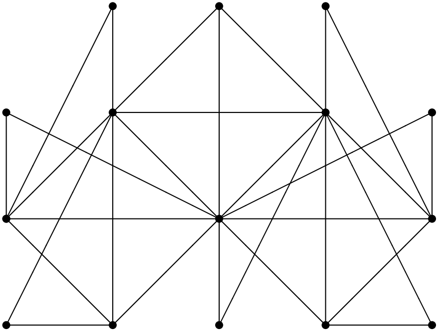
Zero-one k-law holds.

EHR$(G, H, 4)$



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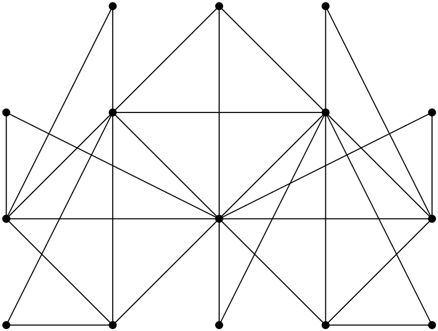
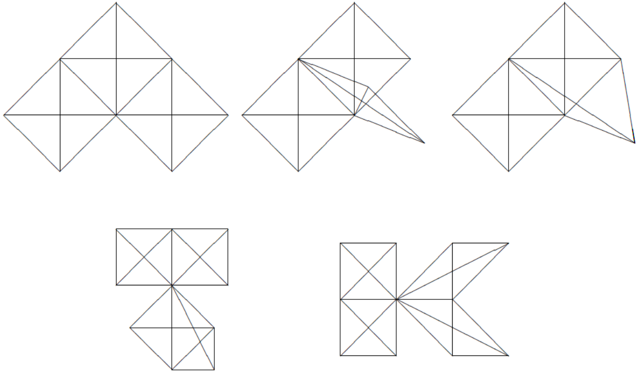
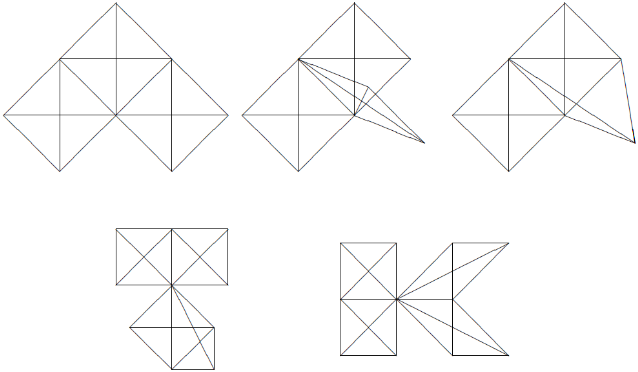


G contains an isomorphic copy X of this graph, $G \setminus X$ doesn't contain any vertex x adjacent to two vertices of X in G.

Probability of such property tends to $\xi \in (0, 1)$.

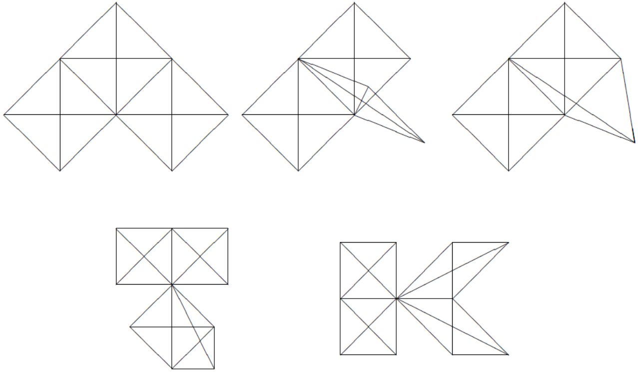


H doesn't contain a copy of any of these graphs.



H doesn't contain a copy of any of these graphs.

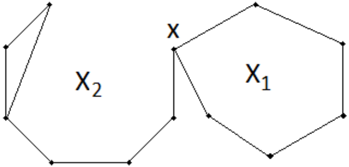
Probability of such property for rather large N greater than $\frac{1}{4}$.



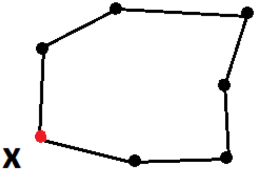
H doesn't contain a copy of any of these graphs.

Probability of such property for rather large N greater than $\frac{1}{4}$.

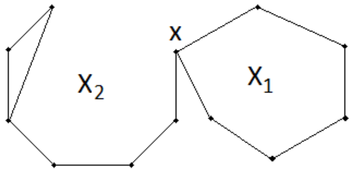
For rather large N Duplicator doesn't have a winning strategy with probability greater than $\frac{\xi}{4}$.



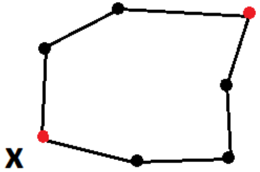
$G(N, N^{-13/14})$ doesn't contain a copy of this graph with probability tending to $\xi \in (0, 1)$. Let G contain $X_1 \cup X_2$, H don't contain $X_1 \cup X_2$.



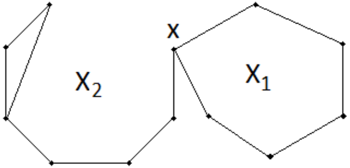
D. chooses y_1, but H doesn't contain (X_1, x)-extension of y_1.



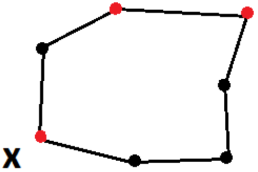
$G(N, N^{-13/14})$ doesn't contain a copy of this graph with probability tending to $\xi \in (0,1)$. Let G contain $X_1 \cup X_2$, H don't contain $X_1 \cup X_2$.



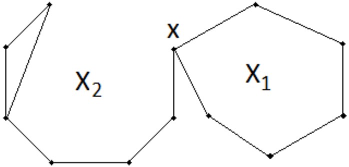
D. chooses y_2, but there is no path with length 3.



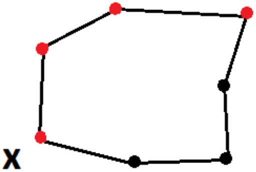
$G(N, N^{-13/14})$ doesn't contain a copy of this graph with probability tending to $\xi \in (0,1)$. Let G contain $X_1 \cup X_2$, H don't contain $X_1 \cup X_2$.



D. chooses y_3.



$G(N, N^{-13/14})$ doesn't contain a copy of this graph with probability tending to $\xi \in (0,1)$. Let G contain $X_1 \cup X_2$, H don't contain $X_1 \cup X_2$.



D. chooses y_4 and S. wins.

Thank you!