Lattice Dimension, Dynamics and Complexity in Symbolic Spaces

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Symbolic spaces

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- **the “lattice”**
  - a semi-group or a group
  - finitely generated
  - *typically:* $\mathbb{Z}^d$

Pro-discrete topology

- configuration

$$c : L \to Q$$

- finite pattern

$$\rho : D \subseteq L \to Q$$

- cylinder set (**basis of the topology**)

$$C_\rho = \{c : \forall z \in D, c(z) = \rho(z)\}$$

- **distance giving the same topology**

$$d(c, c') = 2^{-\min\{|z|:c(z)\neq c'(z)\}}$$

Q^L is compact

This talk

- **objects:** systems defined by local interactions
  - subshift of finite type
  - cellular automata

How does the dynamics and complexity change when we change the lattice?

Subshifts

- action of L on configurations: shift σ_z

$$\sigma_z(c) = z' \mapsto c(z + z')$$

- **subshift** = closed shift-invariant set
- equivalent definition by forbidden language

$$\Sigma_X = \{c \in Q^L : \mathcal{L}(c) \cap X = \emptyset\}$$

- **S**ubshift of **F**inite **T**ype = Σ_X with X finite

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Theorem (Berger 1966)

Aperiodic SFT exist and the Domino Problem is undecidable.

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- extensions to some other lattices exist

Cellular Automata

- *Syntactical object*
  - **neighborhood**: a finite domain D
  - **local rule**: $f : Q^D \to Q$
- *Dynamical system*
  - global function : $F : Q^L \to Q^L$ s.t.

$$F(c)_z = f(c_{[D,z]})$$

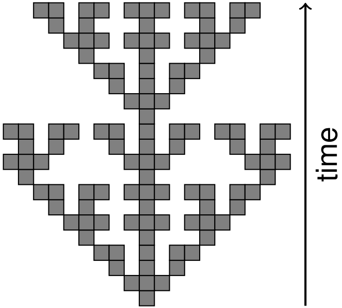
where $c_{[D,z]}$ is the finite pattern : $z' \in D \mapsto c(z + z')$

Hedlund's Theorem

F is a cellular automaton **iff** it is continuous and shift invariant.

Example: Sum mod 2

- $L = \mathbb{Z}$
- $Q = \{0, 1\}$
- $D = \{-1, 0, 1\}$
- $f(x, y, z) = x + y + z \bmod 2$



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- **examples**:
  - $\exists x, F(x) = x$ (**having a fixed point**)
  - $\forall y, \exists x, F(x) = y$ (**being surjective**)
  - $\forall x, \forall y, F(x) = F(y) \Rightarrow x = y$ (**being injective**)

Bounded Time Properties in CA

$L = \mathbb{Z}$

- fix some bounded time property $\mathcal{P}$
- what is the complexity of the following problem:

input	the local definition of a CA F
question	does F verify $\mathcal{P}$?

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Theorem (Sutner, 2009)

When the lattice is $\mathbb{Z}$, any bounded time property is decidable.

- ω-automatic structures have a decidable first-order theory (Büchi 60s and Hodgson 1983)

Bounded Time Properties in CA

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Open

- what is decidable in 2D?
- differences between 2D and higher-D?
- which properties are decidable for any lattice?

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- Moore-Myhill theorem: **surjectivity** $\Leftrightarrow$ **pre-injectivity**

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- Question: different tautologies for 1D and higher-D?

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  3. quantification over configurations and integers.
- **Eq** $\equiv$ **having an equicontinuous point**

$$\exists \mathbf{x}, \forall n, \exists m, \forall \mathbf{y}, \forall t,\ d(\mathbf{x},\mathbf{y}) \leq \frac{1}{m} \Rightarrow d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) \leq \frac{1}{n}$$

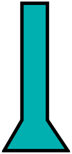
- **S** $\equiv$ **sensitive to initial conditions**

$$\exists n, \forall \mathbf{x}, \forall m, \exists \mathbf{y}, \exists t,\ d(\mathbf{x},\mathbf{y}) \leq \frac{1}{m} \text{ and } d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) > \frac{1}{n}$$

Topological Dynamics in CA

$L = \mathbb{Z}$, the realm of information walls

- F a CA with radius r
  - Obstacle



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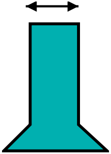

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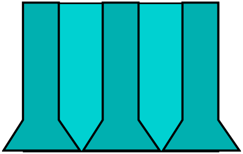

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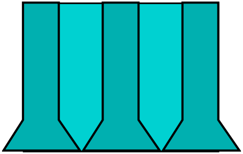

Theorem (Kůrka, 1997)

1. $F \in \mathbf{S} \iff F$ has no wall $\iff F \notin \mathbf{Eq}$
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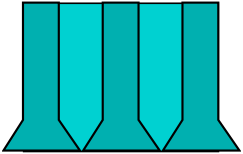

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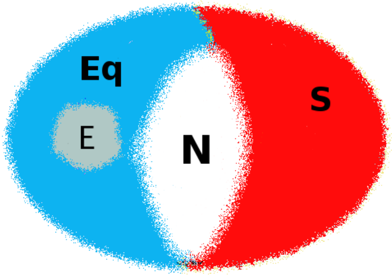
- **intuition:** equicontinuous point = infinite concatenation of walls...

Topological Dynamics in CA

$L = \mathbb{Z}^d$ **with** $d \geq 2$

Theorem (Sablik-Theyssier, 2008)

1. there exists F outside $\mathbf{S} \cup \mathbf{Eq}$
2. sensitivity constant is uncomputable for $F \in \mathbf{S}$
3. $\exists F \in \mathbf{Eq}$ having only non-recursive equicontinuity points



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The set **S** of sensitive CA is

- arithmetical at most at level 2 for $L = \mathbb{Z}$
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Proposition

The set **S** is Σ^0_3 for any lattice.

Arithmetical or not?

- $\phi(X, t)$ **uniformly recursive** if there is a recursive b s.t.

$$\forall X, \forall t : |\phi(X, t)| \leq b(t)$$

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Compacity lemma

$\exists X, \forall t, \phi(X, t)$ can be rewritten to an equivalent Π^0_1 formula

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Theorem (Harel, 1986)

For $L = \mathbb{Z}^2$, the set of CA having a fixed point with infinitely many 0s is not arithmetical

$$\exists X, \forall t, \exists t', \phi(X, t, t')$$

Some Questions

- non-arithmetical proporties in topological dynamics? **Eq**?
- bounded time properties are all arithmetical in any lattice?
- what about arithmetical level of **S** for $\mathbb{Z}$ and $\mathbb{Z}^2$?
- a property with unbounded arithmetical level when lattice changes?
- Turing jump with dimension?

Some other topics

1. complexity of predicting majority vote CA is
   - NC in 1D
   - P-complete in 3D
   - unknown in 2D
2. complexity of testing conjugacy of 2 SFT is
   - decidable for $L = \mathbb{N}$
   - Σ_1^0-complete in 2D
   - unknown for $L = \mathbb{Z}$
3. computational power of self-assembly tilings at temp. 1
   - any Turing-computation in 3D
   - unknown for 2D

Thank you!