α-null sets: strong and weak

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Classical measure theory

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- Hausdorff dimension of set X: the infimum of α such that X is α-null.

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- Main theorem (Martin-Löf) shows the difference between (classical) null sets and effectively null sets: there exists a maximum effectively null set.

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- An other trick: if we have first interval I_1 and then $I_2 \supset I_1$, replace I_2 by $I_2 \setminus I_1$ (= a finite union of disjoint intervals).
- No problems for effective null sets.

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- In non-effective case there is no difference between - we can leave only maximal intervals.
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- In effective case these two definitions are non-equivialent.

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- Similar result from algorithmic randomness (Solovay randomness criterion): X is an effectively null set iff there exists a computabely enumerable set of intervals with finite sum of sizes covering every point of X infinitely many times.

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- Similar result from algorithmic randomness (Solovay randomness criterion): X is an effectively null set iff there exists a computabely enumerable set of intervals with finite sum of sizes covering every point of X infinitely many times.
- Similary we can define Solovay α-null sets: X is a *Solovay effectively null set* if there exists computable enumerable set of intervals with finite sum of α-sizes that covers every point in X infinitely many times.

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Effective α-null sets: final picture

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- Next class - Solovay α-null sets. There is no maximum Solovay α-null set.
- The biggest class: weak α-null sets. Again there exists maximum weak null set.

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- Stronger result about difference between weak and strong α-null sets: there exists a weak effectively α-null set with strong effective size 1. (Every c.e. set of intervals covering this set has sum of α-sizes at least 1).

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- Max measure $\mu(\cup I_k)$, for all μ in some class (class of α-capacitable measures: measures such that for all intervals size determined by this measure is less then its α-size).

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In this game Alice can always win.